Math 115A, Lecture 2
Linear Algebra

Midterm 1

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

Decide whether each of the following sets V with the operations of addition and scalar multiplication specified is a vector space. Justify your answers.

(a) [5pts.] $V \subset \mathrm{Mat}_{2\times 2}(\mathbb{R})$ is the set of 2×2 matrices with determinant zero, and the operations inherited from $\mathrm{Mat}_{2\times 2}(\mathbb{R})$. (Recall that if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 matrix, the determinant is $ad - bc$.)

Solution: Notice that this set is not preserved by addition:

$$\det\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \det\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0, \text{ but} \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

Ergo it is not a subspace of $\mathrm{Mat}_{2\times 2}(\mathbb{R})$, and in particular not a vector space.

(b) [5pts.] $V = \{(a_, a_2) : a_1, a_2 \in \mathbb{R}\}$ with operations

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 - 3b_2)$$
$$c(a_1, a_2) = (ca_1, c^2 a_2)$$

Solution: V is not a vector space because (e.g.) addition fails to commute. Notice that $(0, 0) + (1, 1) = (2, -3)$, but $(1, 1) + (0, 0) = (1, 1)$.

Problem 2.

Consider the subset V of polynomials $P_2(\mathbb{R})$ such that, for any $ax^2 + bx + c$ in V, we have $a + b + c = 0$.

(a) [5pts.] Prove that V is a subspace of $P_2(\mathbb{R})$.

Solution:

- The additive identity, $0x^2 + 0x + 0$, is in V.
- If $ax^2 + bx + c$ and $ex^2 + fx + g$ are in V, then $a + b + c = 0 = e + f + g$. Hence their sum $(a+e)x^2+(b+f)x+(g+c)$ has $(a+e)+(b+f)+(g+c) = (a + b + c) + (d + e + f) = 0$, and is in V.
- If $ax^2 + bx + c \in V$, then $a + b + c = 0$. So if $h \in \mathbb{R}$, the scalar product $h(ax^2+bx+c) = (ha)x^2+(hb)x+(hc)$ has $ha+hb+hc = h(a+b+c) = 0$, and is in V.

(b) [5pts.] Find the dimension of V.

Solution: We see that any ax^2+bx+c in V can be rewritten $ax^2+bx+(-a-b) = a(x^2-1)+b(x-1)$. Therefore $\beta = \{x^2-1, x-1\}$ spans V, and is also clearly linearly independent since its two elements have different degrees, so β is a basis for V. Hence the dimension of V is two.

Problem 3.

Consider the set $S = \{(2,3,5),(1,0,-1),(-2,1,7),(1,4,11)\} \subset \mathbb{R}^3$.

(a) [5pts.] Is S linearly independent or linearly dependent? Justify your answer *without doing a computation.*

Solution: The dimension of $\mathbb{R}^3$ is three, so any linearly independent set in $\mathbb{R}^3$ must have no more than three elements. Ergo S must be linearly dependent.

(b) [5pts.] Find a subset of S that is a basis for $\mathbb{R}^3$.

Solution: We build a maximal linearly independent subset for $\mathbb{R}^3$. First, $\{(2,3,5)\}$ is linearly independent because it is a set consisting of a single nonzero vector. Next, $\{(2,3,5),(1,0,-1)\}$ is linearly independent because neither vector is a multiple of the other. Finally, consider $\beta = \{(2,3,5),(1,0,-1),(-2,1,7)\}$. If some linear combination $a(2,3,5)+b(1,0,-1)+c(-2,1,7) = 0$, we have

$$\begin{cases} 2a+b-2c = 0 \\ 3a+c = 0 \\ 5a-b+7c = 0 \end{cases}$$

In any nontrivial solution, we must have $c = 0$ (because the set $\{(2,3,5),(1,0,-1)\}$ is linearly independent) so after possibly scaling we can assume $c = 1$. Hence our equations become

$$\begin{cases} 2a+b = 2 \\ 3a = -1 \\ 5a-b = -7 \end{cases}$$

The second equation gives $a = -13$, so by the first equation $b = \frac{8}{3}$.But then the last equation becomes $-\frac{13}{3} = -7$. So no nontrivial solution exists. Hence β is linearly independent and, having three elements, is a basis for $\mathbb{R}^3$.

Problem 4.

Let S_1 and S_2 be subsets of a vector space V.

(a) [5pts.] Prove that $\text{span}(S_1 \cap S_2) \subset \text{span}(S_1) \cap \text{span}(S_2)$.

Solution: Suppose that $v \in \text{span}(S_1 \cap S_2)$, so $v = a_1u_1 + \cdots + a_nu_n$ is a linear combination of vectors $u_1, \cdots, u_n \in S_1 \cap S_2$. Then since each u_i is an element of S_1, v is also a linear combination of elements of S_1, hence $v \in \text{span}(S_1)$. Similarly, $v \in \text{span}(S_2)$. So $v \in \text{span}(S_1) \cap \text{span}(S_2)$. Since v was arbitrary, $(S_1 \cap S_2) \subset \text{span}(S_1) \cap \text{span}(S_2)$.

(b) [5pts.] Give an example in which the sets above are equal and one in which they are unequal.

Solution: For equality, consider $S_1 = \{(1,0),(0,1)\}$ and $S_2 = \{(1,0\}$ in $\mathbb{R}^2$. Then $\text{span}(S_1) = \mathbb{R}^2$, and $\text{span}(S_2)$ is the x-axis. Then $\text{span}(S_1) \cap \text{span}(S_2)$ is the x-axis as well, so since $S_1 \cap S_2 = \{(1,0)\}$, $(S_1 \cap S_2) = \text{span}(S_1) \cap \text{span}(S_2)$. For inequality, let $S_1 = \{(1,0),(0,1)\}$ in $\mathbb{R}^2$ and $S_2 = \{(1,1)\}$. We see that $\text{span}(S_1 \cap S_2) = \text{span}(\phi) = \{0\}$. But $\text{span}(S_1) \cap \text{span}(S_2) = \mathbb{R}^2 \cap \text{span}(S_2) = \text{span}(S_2)$ is the line $\text{span}(\{(1,1)\})$.

Problem 5.

Recall that if W_1 and W_2 are subspaces of a vector space V, then

$$W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}.$$

If in addition $W_1 \cap W_2 = \emptyset$, then we call this space $W_1 \oplus W_2$. If $W_1 \oplus W_2 = V$, then W_2 is said to be the complement of W_1.

(a) [5pts.] Prove that the xy-plane and the z-axis are complements in $\mathbb{R}^3$.

Solution: The xy-plane is the subspace $\{(x,y,0) : x,y \in \mathbb{R}$ and the z-axis is the subspace $\{(0,0,z) : z \in \mathbb{R}\}$. These subspaces certainly have intersection $\{(0,0,0)\}$, and moreover any $(x,y,z) \in \mathbb{R}^3$ may be expressed as $(x,y,0) + (0,0,z)$.

(b) [5pts.] Let V be an n-dimensional vector space, and W_1 a k-dimensional subspace of V. Prove that W_1 has a complement; that is, prove that there exists W_2 such that $W_1 \oplus W_2 = V$. [Hint: Start with a basis for W_1, and extend to a basis for V. Now you should be able to find a candidate basis for W_2.]

Solution: Let $\{x_1, \cdots, x_k\}$ be a basis for W_1, and extend to a basis $\beta = \{x_1, \cdots, x_k, y_1, \cdots, y_{n-k}\}$ for V. Then let $W_2 = \text{span}(\{y_1, \cdots, y_{n-k}\})$. We claim that $V = W_1 + W_2$. For any $v \in V$ is a linear combination of elements of β, and therefore may be written $a_1x_1 + \cdots a_kx_k + b_1y_1 + \cdots b_{n-k}y_{n-k} = (a_1x_1 + \cdots a_kx_k) + (b_1y_1 + \cdots b_{n-k}y_{n-k})$, a sum of elements in $W_1 = \text{span}(\{x_1, \cdots, x_k\})$ and $W_2 = \text{span}(\{y_1, \cdots, y_{n-k}\})$. Ergo $V = W_1 + W_2$. Moreover, suppose that $v \in W_1 \cap W_2$. Then we may write $v = a_1x_1 + \cdots + a_kx_k$, because $v \in W_1$, but also $v = b_1y_1 + \cdots + b_{n-k}y_{n-k}$, because $v \in W_2$. So $0 = a_1x_1 + \cdots + a_kx_k - b_1y_1 - \cdots - b_{n-k}y_{n-k}$. Since β is a basis, this implies all of the a_i and b_j are in fact zero, and we conclude that $v = 0$. So $W_1 \cap W_2 = \{0\}$, and we see $V = W_1 \oplus W_2$.